

VICIOUS WALKERS AND RANDOM CONTRACTION MATRICES

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ABSTRACT. The ensemble $\text{CUE}^{(q)}$ of truncated random unitary matrices is a deformation of the usual Circular Unitary Ensemble depending on a discrete non-negative parameter q . $\text{CUE}^{(q)}$ is an exactly solved model of random contraction matrices originally introduced in the context of scattering theory. In this article, we exhibit a connection between $\text{CUE}^{(q)}$ and Fisher's random-turns vicious walker model from statistical mechanics. In particular, we show that the moment generating function of the trace of a random matrix from $\text{CUE}^{(q)}$ is a generating series for the partition function of Fisher's model, when the walkers are assumed to represent mutually attracting particles.

1. INTRODUCTION

1.1. Truncated random unitary matrices. Fix an integer $d \geq 1$, and let $\mathbb{M}_d = \mathbb{M}_d(\mathbb{C})$ be the space of $d \times d$ complex matrices. Consider the linear map

$$(1) \quad T : \mathbb{M}_{d+1} \rightarrow \mathbb{M}_d$$

which acts by removing the last row and column of a matrix. For example, $T : \mathbb{M}_3 \rightarrow \mathbb{M}_2$ is defined by

$$(2) \quad T \left(\begin{bmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{bmatrix} \right) = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix}$$

T is called the *truncation* map. Since $\|T(M)\| \leq \|M\|$ in operator norm for any $M \in \mathbb{M}_{d+1}$, T maps the unitary group

$$(3) \quad \mathbb{U}_{d+1} = \{U \in \mathbb{M}_{d+1} : U^* = U^{-1}\}$$

into

$$(4) \quad \mathbb{B}_d = \{P \in \mathbb{M}_d : \|P\| \leq 1\},$$

the semigroup of linear contractions of Euclidean space \mathbb{C}^d . More generally, for any integer $q \geq 0$, the q -fold composition $T^{(q)}$ maps each matrix in \mathbb{M}_{d+q} to its $d \times d$ principal submatrix and thus sends \mathbb{U}_{d+q} into \mathbb{B}_d .

The truncation operator induces a very natural deformation of the Circular Unitary Ensemble (CUE) from random matrix theory. Consider the unitary group \mathbb{U}_{d+q} as a probability space, with the Borel σ -algebra and Haar probability measure. The pushforward $\gamma_d^{(q)}$ of Haar measure on \mathbb{U}_{d+q} under $T^{(q)}$ is a Borel probability measure on \mathbb{B}_d , and thus one obtains a random matrix ensemble

$$(5) \quad \{(\mathbb{B}_d, \gamma_d^{(q)}) : d \geq 1\}$$

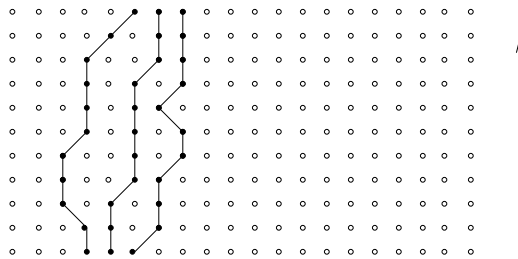


FIGURE 1. *Three random-turns particles moving between ground states.*

which will be denoted $\text{CUE}^{(q)}$ and called an ensemble of *truncated random unitary matrices*. Since $T^{(0)}$ is the identity operator on \mathbb{M}_d , $\text{CUE}^{(0)}$ reduces to the usual Circular Unitary Ensemble of unitary matrices under Haar measure when $q = 0$.

Ensembles of truncated random unitary matrices were first studied by Sommers and Zyczkowski [35] in the context of quantum chaotic scattering, and have been further investigated by Petz and Réffy [31], [32], Fyodorov and Khoruzhenko [17], and Neretin [28]. They are an important technical ingredient in Krishnapur's recent study of random matrix-valued analytic functions [25].

Certain averages over CUE are well-known to be related to the combinatorics of increasing subsequences in permutations [33], [4] and vicious walkers on \mathbb{Z} [1], [15]. A natural question is whether (and in what form) this relationship with combinatorics extends to the deformed ensemble $\text{CUE}^{(q)}$.

1.2. Fisher's random-turns model. Vicious walker models were introduced in statistical mechanics by Fisher in order to model wetting and melting [12]. Fisher's *random-turns* vicious walker model consists of a system of d particles ("walkers") initially occupying sites

$$(6) \quad \mu = \mu_1 > \mu_2 > \cdots > \mu_d$$

on the integer lattice \mathbb{Z} . Note that the walkers are labelled from right to left, so that we list their positions in decreasing order. The system evolves in discrete time according to the following rule: at each instant, a single random particle makes a random unit jump left or right (a "random turn"), subject only to the constraint that no two particles can occupy the same lattice site simultaneously. The function $Z_d(N; \mu, \lambda)$ which counts the number of ways in which d random-turns particles can depart initial sites μ and arrive at new sites

$$(7) \quad \lambda = \lambda_1 > \lambda_2 > \cdots > \lambda_d$$

at time N is the *partition function* of the model.

If one assumes the existence of an attractive force between the particles, then the system can be at equilibrium only when the particles are on adjacent sites. In this case we use the simplified notation $Z_d(N; q)$ to denote the number of ways in which d random-turns particles can move between ground states q sites apart in N instants. Figure 1 gives an example of a sequence of configurations of mutually attracting random-turns particles counted by $Z_3(10; 2)$.

1.3. Main result. Our main result is the following. Consider the matrix integral

$$(8) \quad G_d(x; q) := \frac{x^{dq}}{H_{d \times q}} \int_{\mathbb{B}_d} e^{x \operatorname{Tr}(P+P^*)} \gamma_d^{(q)}(dP),$$

where

$$(9) \quad H_{d \times q} = \prod_{i=0}^{d-1} \frac{(q+i)!}{i!}$$

is the hook-product of the $d \times q$ rectangular Young diagram. $G_d(x; q)$ is a scaled and shifted version of the moment generating function of $\operatorname{Tr}(P_d^{(q)} + P_d^{(q)*})$, where $P_d^{(q)}$ is a random matrix from $\operatorname{CUE}^{(q)}$.

Theorem 1.1. $G_d(x; q)$ is the exponential generating series for the partition function of a system of d mutually attracting random-turns particles:

$$(10) \quad G_d(x; q) = \sum_{N \geq 0} Z_d(N; q) \frac{x^N}{N!}.$$

Below, we use this interpretation of $G_d(n; q)$ as a generating series to give a completely combinatorial proof of the following identity, which expresses the matrix integral $G_d(x; q)$ as a Toeplitz determinant of Bessel functions.

Theorem 1.2. We have

$$(11) \quad G_d(x; q) = \det(I_{q+j-i}(2x))_{1 \leq i, j \leq d},$$

where the entries in the determinant are modified Bessel functions.

Remark: Neretin [28] has shown that, when $q \geq d$, the measure $\gamma_d^{(q)}$ becomes absolutely continuous with respect to Lebesgue measure on \mathbb{B}_d and has density

$$(12) \quad \frac{H_{d \times q}}{\pi^{d^2} H_{d \times (q-d)}} \det(I - P^*P)^{q-d} dP.$$

Therefore when $q \geq d$, $G_d(x; q)$ can be written more concretely as

$$(13) \quad G_d(x; q) = \frac{x^{dq}}{\pi^{d^2} H_{d \times (q-d)}} \int_{\mathbb{B}_d} e^{x \operatorname{Tr}(P+P^*)} \det(I - P^*P)^{q-d} dP.$$

1.4. Connection with the increasing subsequence problem. Recall that a permutation σ from the symmetric group $S(n)$ is said to have an *increasing subsequence* of length k if there exist indices

$$(14) \quad 1 \leq i_1 < i_2 < \cdots < i_k \leq n$$

such that

$$(15) \quad \sigma(i_1) < \sigma(i_2) < \cdots < \sigma(i_k).$$

Increasing subsequences in permutations were first studied by Erdős and Szekeres [11] in the 1930s in connection with a Ramsey-type problem for the permutation group. In the 1960s, Ulam [40] raised the problem of determining the number $u_d(n)$ of permutations in $S(n)$ with increasing subsequence length bounded by d . This came to be known as the *increasing subsequence problem*. Stanley's ICM contribution [38] gives a comprehensive survey, from a combinatorial perspective, of the vast literature on increasing subsequences in permutations and related topics.

Forrester [14] has observed the following connection between the increasing subsequence problem and Fisher's random-turns model:

$$(16) \quad Z_d(N; 0) = \begin{cases} \binom{2n}{n} u_d(n), & \text{if } N = 2n \text{ for some } n \geq 0 \\ 0, & \text{otherwise} \end{cases}.$$

We will present a new proof of (a more general version of) Forrester's result below. Our approach is based on the fact that the configuration space of Fisher's random-turns model naturally carries the structure of a graded graph with commutative raising and lowering operators. This approach was inspired by Stanley's notion of a "differential poset," defined as a graded graph which arises as the Hasse graph of a poset and whose raising and lowering operators satisfy the Heisenberg commutation relation [36]. In particular, this observation implies that specializing $q = 0$ in Theorems 1.1 and 1.2, we obtain

$$(17) \quad G_d(x; 0) = \sum_{n \geq 0} u_d(n) \frac{x^{2n}}{n!n!} = \det(I_{j-i}(2x))_{1 \leq i, j \leq d}.$$

The identity between the matrix integral and the generating series of $u_d(n)$ is due to Rains [33], while the identity between the series and the determinant is due to Gessel [18]. Both were discovered in the context of the enumeration of permutations with bounded increasing subsequence length.

2. CONFIGURATION SPACE AS A GRADED GRAPH

2.1. Configuration space of random-turns particles. The configuration space of a physical system is the set of all possible positions of its constituents. In the case of the random-turns model, this is the discrete space

$$(18) \quad \mathcal{W}_d = \{(\lambda_1, \lambda_2, \dots, \lambda_d) \in \mathbb{Z}^d : \lambda_1 > \lambda_2 > \dots > \lambda_d\}.$$

\mathcal{W}_d is familiar as a type A *Weyl lattice*, i.e. it is the intersection of \mathbb{Z}^d with an open Weyl chamber for the type A root system in \mathbb{R}^d (see e.g. [8] about root systems and their Weyl chambers). \mathcal{W}_d becomes a simple, connected, locally finite graph when we declare vertices $\mu, \lambda \in \mathcal{W}_d$ adjacent if and only if they are unit Euclidean distance apart. The partition function $Z_d(N; \mu, \lambda)$ of the random-turns model counts the number of walks of length N from μ to λ on the graph \mathcal{W}_d .

\mathcal{W}_d is also an example of a *graded graph*. For our purposes, it suffices to define a graded graph to be a pair (G, r) consisting of a simple, connected, locally finite graph G together with a function $r : G \rightarrow \mathbb{Z}$, called the *rank function*, which associates an integer to each vertex of G . The incidence relation and the rank function on G are required to be compatible in the sense that if $u, v \in G$ are adjacent vertices, then either $r(v) = r(u) + 1$ (denoted $u \nearrow v$) or $r(v) = r(u) - 1$ (denoted $u \searrow v$). Graded graphs occur frequently in combinatorics as the Hasse graphs of posets (see [36] and [13]) and in representation theory where they are known as *branching graphs* or *Bratteli diagrams* (see e.g. [24]).

If we define a rank function r on \mathcal{W}_d by

$$(19) \quad r(\lambda) = r(\lambda_1, \dots, \lambda_d) := \sum_{i=1}^d \lambda_i - \binom{d+1}{2},$$

then (\mathcal{W}_d, r) is a graded graph. Note that we have normalized the rank function in such a way that the rank of the canonical ground state $\rho = (d, d-1, \dots, 1)$ in Fisher's model is 0.

Let (G, r) be a graded graph. By the *unrefined* partition function of G we mean the number $Z_G(N; u, v)$ of walks on G from u to v . We will refer to the number $Z_G(L^{b_k} R^{a_k} \dots L^{b_1} R^{a_1}; u, v)$ of walks on G from u to v of the form

$$(20) \quad u \underbrace{\nearrow \nearrow \dots \nearrow \searrow \searrow \dots \searrow}_{a_1} \dots \underbrace{\searrow \searrow \dots \searrow \nearrow \nearrow \dots \nearrow}_{b_1} \dots \underbrace{\nearrow \nearrow \dots \nearrow \searrow \searrow \dots \searrow}_{a_k} \underbrace{\searrow \searrow \dots \searrow \nearrow \nearrow \dots \nearrow}_{b_k} v$$

as the *refined* partition function of G . In the special case $G = \mathcal{W}_d$ we will continue to use the notation $Z_d(N; \mu, \lambda) = Z_{\mathcal{W}_d}(N; \mu, \lambda)$ from the Introduction, as well as $Z_d(L^{b_k} R^{a_k} \dots L^{b_1} R^{a_1}; \mu, \lambda) := Z_{\mathcal{W}_d}(L^{b_k} R^{a_k} \dots L^{b_1} R^{a_1}; \mu, \lambda)$.

L and R as defined above are simply inert symbols recording instances of positive and negative steps on the configuration graph (equivalently left and right jumps of the particles). We may represent L and R as linear operators acting on $\mathbb{C}[G]$, the free \mathbb{C} -vector space spanned by the vertices of G , by defining

$$(21) \quad R(u) := \sum_{u \nearrow v} v \text{ and } L(u) := \sum_{u \searrow w} w$$

and extending linearly over $\mathbb{C}[G]$. R and L are then called the *raising* and *lowering* operators on G . The monoid $\{L, R\}^*$ generated by the raising and lowering operators determines the refined partition function, since

$$(22) \quad Z_G(L^{b_k} R^{a_k} \dots L^{b_1} R^{a_1}; u, v) = [v] L^{b_k} R^{a_k} \dots L^{b_1} R^{a_1}(u)$$

where $[v]$ is the “coefficient of v ” functional on $\mathbb{C}[G]$.

Theorem 2.1. *The raising and lowering operators on (\mathcal{W}_d, r) commute.*

Proof. Let $\overline{\mathcal{W}}_d$ be the graded graph with vertex set

$$(23) \quad \overline{\mathcal{W}}_d = \{(\lambda_1, \lambda_2, \dots, \lambda_d) \in \mathbb{Z}^d : \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_d\},$$

incidence relation as with \mathcal{W}_d , and rank function defined by

$$(24) \quad \bar{r}(\lambda) = \bar{r}(\lambda_1, \lambda_2, \dots, \lambda_d) := \sum_{i=1}^r \lambda_i.$$

Since the translation

$$(25) \quad \lambda \mapsto \bar{\lambda} := \lambda - \rho$$

is a rank-preserving isomorphism of \mathcal{W}_d with $\overline{\mathcal{W}}_d$, it suffices to prove that the raising and lowering operators associated to $(\overline{\mathcal{W}}_d, \bar{r})$ commute.

Let $\mu, \lambda \in \overline{\mathcal{W}}_d$ be arbitrary vertices. Consider separately the cases $\mu \neq \lambda$ and $\mu = \lambda$.

Case $\mu \neq \lambda$: $[\lambda]LR(\mu)$ counts the number of walks on \mathcal{W}_d from μ to λ of the form

$$(26) \quad \mu \nearrow \mu + e_i \searrow \mu + e_i - e_j = \lambda,$$

where e_i and e_j are standard basis vectors of \mathbb{R}^n . If such an i and j exist, then $i \neq j$ by the assumption $\mu \neq \lambda$. Thus

$$(27) \quad \mu \searrow \mu - e_j \nearrow \mu - e_j + e_i$$

is a valid walk on \mathcal{W}_d from μ to λ , and we have a bijection

$$(28) \quad \mu \nearrow v \searrow \lambda \iff \mu \searrow v' \nearrow \lambda$$

between up-down walks on \mathcal{W}_d from μ to λ and down-up walks on \mathcal{W}_d from μ to λ . Hence $[\lambda]LR(\mu) = [\lambda]RL(\mu)$.

Case $\mu = \lambda$: In this case, let r be the number of distinct entries of $\mu = \lambda$, i.e. $i_1 < i_2 < \dots < i_r$ satisfy $\mu_{i_1} > \mu_{i_2} > \dots > \mu_{i_r}$. Then it is clear that $[\mu]RL(\mu) = [\mu]LR(\mu) = r$.

□

2.2. Determinant identities. Using Theorem 2.1, we can easily prove determinantal identities for the refined partition function of random-turns particles. Let I_k be the modified Bessel function of order k ,

$$(29) \quad I_k(2x) := \sum_{n \geq 0} \frac{x^n}{\Gamma(n+1)} \frac{x^{n+k}}{\Gamma(n+k+1)}.$$

It is known that $I_{-k}(2x) = I_k(2x)$ for $k \in \mathbb{Z}$ (see e.g. [7] about Bessel functions and their properties).

Proposition 2.2. *Let $\mu, \lambda \in \mathcal{W}_d$ be configurations with $r(\mu) \leq r(\lambda)$, and let $W_0, W_1, \dots, W_n, \dots \in \{L, R\}^*$ be any sequence of words in the raising and lowering operators on \mathcal{W}_d verifying*

$$(30) \quad \deg_L W_n = n \text{ and } \deg_R W_n = n + r(\lambda) - r(\mu).$$

Then

$$(31) \quad \sum_{n \geq 0} Z_d(W_n; \mu, \lambda) \frac{x^{2n+r(\lambda)-r(\mu)}}{n!(n+r(\lambda)-r(\mu))!} = \det(I_{\lambda_i-\mu_j}(2x))_{1 \leq i, j \leq d}.$$

Proof. Since \mathcal{W}_d is the intersection of \mathbb{Z}^d with an open type A Weyl chamber in \mathbb{R}^d , it follows immediately from the André-Gessel-Zeilberger reflection principle [20], [21], [19] that the generating series for the unrefined partition function is

$$(32) \quad \sum_{N \geq 0} Z_d(N; \mu, \lambda) \frac{x^N}{N!} = \det(I_{\lambda_i-\mu_j}(2x))_{1 \leq i, j \leq d}.$$

Now, an N -step walk from μ to λ on \mathcal{W}_d exists if and only if $N = 2n + r(\lambda) - r(\mu)$ for some $n \geq 0$ (this n being the number of negative steps). By Theorem 2.1, the number of such walks is

$$(33) \quad \binom{2n+r(\lambda)-r(\mu)}{n} Z_d(W_n; \mu, \lambda)$$

for any $W_n \in \{L, R\}^*$ with $\deg_L W_n = n$ and $\deg_R W_n = n + r(\lambda) - r(\mu)$. Thus

$$(34) \quad \sum_{N \geq 0} Z_d(N; \mu, \lambda) \frac{x^N}{N!} = \sum_{n \geq 0} \binom{2n+r(\lambda)-r(\mu)}{n} Z_d(W_n; \mu, \lambda) \frac{x^{2n+r(\lambda)-r(\mu)}}{(2n+r(\lambda)-r(\mu))!}$$

$$(35) \quad = \sum_{n \geq 0} Z_d(W_n; \mu, \lambda) \frac{x^{2n+r(\lambda)-r(\mu)}}{n!(n+r(\lambda)-r(\mu))!}$$

and the result follows. □

Corollary 2.2 generalizes recent results of Xin ([42], Theorem 11 and Proposition 12), who proves the case $W_n = L^n R^n$ using the “Stanton-Stembridge trick.”

2.3. Young tableaux and increasing subsequences. Let \mathbb{Y} be the *Young graph*, i.e. the Hasse graph of the lattice of Young diagrams partially ordered by inclusion of diagrams (see [37]). \mathbb{Y} is a graded graph, where the rank $|\lambda|$ of a Young diagram $\lambda \in \mathbb{Y}$ is the number of cells in λ . \mathbb{Y} has a unique vertex \emptyset of rank 0 (the “empty diagram”), and all other vertices have positive rank. Walks on the Young graph are known as *oscillating Young tableaux*.

Let \mathbb{Y}_d be the induced subgraph of \mathbb{Y} whose vertices are the Young diagrams with at most d rows. It is a well-known consequence of the RSK correspondence (see [37] as well as the original article of Schensted [34]) that

$$(36) \quad Z_{\mathbb{Y}_d}(\mathbb{L}^n \mathbb{R}^n; \emptyset, \emptyset) = u_d(n),$$

where $u_d(n)$ is the number of permutations in $S(n)$ with no increasing subsequence of length greater than d .

Observe that there is a canonical embedding $\iota: \mathbb{Y}_d \hookrightarrow \overline{\mathcal{W}}_d$ obtained by mapping each Young diagram λ in \mathbb{Y}_d onto the vector of its row lengths. Composing ι with the translation

$$(37) \quad \lambda \mapsto \lambda^\circ := \lambda + \rho$$

gives an embedding of \mathbb{Y}_d into \mathcal{W}_d . Thus it is an immediate consequence of Theorem 2.1 that

$$(38) \quad Z_d(N; 0) = \begin{cases} \binom{2n}{n} u_d(n), & \text{if } N = 2n \\ 0, & \text{otherwise} \end{cases}$$

which is precisely Forrester’s result (16). We therefore have the following Proposition as a direct consequence of Proposition 2.2.

Proposition 2.3. *For any $d \geq 1$,*

$$(39) \quad \sum_{n \geq 0} u_d(n) \frac{x^{2n}}{n!n!} = \det(I_{i-j}(2x))_{1 \leq i, j \leq d}.$$

Proof. Choose $\mu = \lambda = \rho$ in Corollary 2.2. □

This is precisely Gessel’s identity, the original proof of which appears in [18]. Alternative proof of this result were later given by Gessel, Weinstein, and Wilf [41], Tracy and Widom [39], and Xin [42]. Gessel’s identity was the starting point of Baik, Deift, and Johansson [3] in their groundbreaking work on the limiting distribution of the length of the longest increasing subsequence in a large random permutation.

3. PROOF OF THE MAIN THEOREM

We now give the proof of Theorem 1.1, which links truncated random unitary matrices with Fisher’s random-turns model. The proof is based on the integral identity

$$(40) \quad G_d(x; q) = \int_{\mathbb{U}_d} e^{x \operatorname{Tr}(U + U^*)} \det(U^*)^q dU,$$

which we deduce from the following remarkable result of Wei and Wettig.

Theorem 3.1. [41] *For any matrices $X, Y \in \mathbb{M}_{(d+q) \times d}$ verifying $\det(Y^*X) \neq 0$ the following holds:*

$$(41) \quad \int_{\mathbb{U}_{d+q}} e^{\text{Tr}(Y^*UX + X^*U^*Y)} dU = H_{d \times q} \int_{\mathbb{U}_d} e^{\text{Tr}(UY^*Y + X^*XU^*)} \det(UY^*X)^{-q} dU.$$

To get (40) from Theorem 3.1, choose

$$(42) \quad X = Y = \begin{bmatrix} t & 0 & \dots & 0 \\ 0 & t & \dots & 0 \\ 0 & 0 & \dots & t \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}$$

where t is a non-zero real number. Then $Y^*X = t^2I$, and $Y^*UX = T^{(q)}(U)$ for any unitary matrix $U \in \mathbb{U}_{d+q}$. Thus Theorem 3.1 reduces to

$$(43) \quad \frac{t^{2dq}}{H_{d \times q}} \int_{\mathbb{B}_d} e^{t^2 \text{Tr}(P+P^*)} \gamma_d^{(q)}(dP) = G_d(t^2; q) = \int_{\mathbb{U}_d} e^{t^2 \text{Tr}(U+U^*)} \det(U^*)^q dU,$$

valid for all real $t \neq 0$. This identity also holds true at $t = 0$, since then the left hand side is obviously equal to 0, and the right hand side is also equal 0 for the following reason:

$$(44) \quad \int_{\mathbb{U}_d} F(U)G(U^*)dU = 0$$

for any two homogeneous polynomials F, G in the entries of U and U^* with $\deg F \neq \deg G$ (this is a standard property of Haar measure, see e.g. [9] for a proof). Thus (43) is true for all real numbers t and we obtain (40).

To complete the proof of Theorem 1.1, we compute the power series expansion of the unitary matrix integral (40) using techniques from symmetric function theory (see [37], Chapter 7). Recall that a monotone walk

$$(45) \quad \emptyset \nearrow \nearrow \dots \nearrow \lambda$$

on the Young graph \mathbb{Y} is called a *standard Young tableau* of shape λ . Following [37], we will use the notation

$$(46) \quad f^\lambda := Z_{\mathbb{Y}}(\mathbb{R}^{|\lambda|}; \emptyset, \lambda)$$

for the number of standard Young tableaux of shape λ .

Expanding the integral (40) as a power series in x , we have

$$(47) \quad G_d(x; q) = \sum_{m, n \geq 0} \int_{\mathbb{U}_d} (\text{Tr } U)^m (\text{Tr } U^*)^n \det(U^*)^q dU \frac{x^{m+n}}{m!n!}.$$

Appealing again to (44), this becomes

$$(48) \quad G_d(x; q) = \sum_{n \geq 0} I_d(n; q) \frac{x^{2n+dq}}{n!(n+dq)!},$$

where

$$(49) \quad I_d(n; q) := \int_{\mathbb{U}_d} (\text{Tr } U)^{n+dq} (\text{Tr } U^*)^n \det(U^*)^q dU.$$

Now we evaluate the integral $I_d(n; q)$. Let Λ_d be the \mathbb{C} -algebra of symmetric polynomials in d indeterminates, and define the *Hall scalar product* on Λ_d by

$$(50) \quad \langle f|g \rangle := \int_{\mathbb{U}_d} f(\mathbf{U})g(\mathbf{U}^*)d\mathbf{U}$$

where $f(\mathbf{U}), g(\mathbf{U}^*)$ denote symmetric polynomials $f, g \in \Lambda_d$ evaluated on the spectra of \mathbf{U} and \mathbf{U}^* respectively. The Hall product is a symmetric, non-negative bilinear form on Λ_d (see [37]). It is well-known that the Schur polynomials $\{s_\lambda\}_{\lambda \in \mathbb{Y}_d}$ constitute a linear basis of Λ_d , and moreover since each map $\mathbf{U} \mapsto s_\lambda(\mathbf{U})$ is the character of an irreducible polynomial representation of \mathbb{U}_d we have

$$(51) \quad \langle s_\lambda | s_\mu \rangle = \delta_{\lambda, \mu}$$

by Schur orthogonality (see e.g. [8]).

Now

$$(52) \quad I_d(n; q) = \langle e_1^{n+dq} | e_1^n e_d^q \rangle,$$

where $e_1, e_d \in \Lambda_d$ are the first and last elementary symmetric polynomials. In terms of Schur functions, one has the linear expansion

$$(53) \quad e_1^N = \sum_{\substack{\lambda \in \mathbb{Y}_d \\ |\lambda| = n}} f^\lambda s_\lambda$$

for powers of the first elementary symmetric polynomial (see [37]). It is also clear from the combinatorial definition of Schur polynomials in terms of semistandard Young tableaux that $e_d^q = s_{d \times q}$. Hence we have

$$(54) \quad I_d(n; q) = \sum_{\substack{\mu, \lambda \in \mathbb{Y}_d \\ |\mu| = n+dq, |\lambda| = n}} f^\mu f^\lambda \langle s_\mu | s_\lambda s_{d \times q} \rangle.$$

The following property of the Hall scalar product is well-known:

$$(55) \quad \langle s_\mu | s_\lambda s_\nu \rangle = \langle s_{\mu/\nu} | s_\lambda \rangle,$$

where $s_{\mu/\nu}$ is a skew Schur polynomial, which is by definition the zero polynomial unless $\mu \supseteq \nu$. In other words, the adjoint of the “multiplication by s_ν ” operator is the “deletion of ν ” operator. Hence we have

$$(56) \quad I_d(n; q) = \sum_{\substack{\mu, \lambda \in \mathbb{Y}_d \\ |\mu| = n+dq, |\lambda| = n}} f^\mu f^\lambda \langle s_{\mu/d \times q} | s_\lambda \rangle = \sum_{\substack{\lambda \in \mathbb{Y}_d \\ |\lambda| = n}} f^{\lambda+d \times q} f^\lambda,$$

where $\lambda + d \times q$ is the concatenation of the Young diagram λ with the rectangular diagram $d \times q$. Thus

$$(57) \quad I_d(n; q) = \sum_{\substack{\lambda \in \mathbb{Y}_d \\ |\lambda| = n}} Z_{\mathbb{Y}_d}(\mathbb{R}^{n+dq}; \emptyset, \lambda + d \times q) Z_{\mathbb{Y}_d}(\mathbb{R}^n; \emptyset, \lambda)$$

$$(58) \quad = \sum_{\substack{\lambda \in \mathbb{Y}_d \\ |\lambda| = n}} Z_{\mathbb{Y}_d}(\mathbb{R}^{n+dq}; \emptyset, \lambda + d \times q) Z_{\mathbb{Y}_d}(\mathbb{L}^n; \lambda + d \times q, d \times q)$$

$$(59) \quad = Z_{\mathbb{Y}_d}(\mathbb{L}^n \mathbb{R}^{n+dq}; \emptyset, d \times q)$$

$$(60) \quad = \binom{2n+dq}{n}^{-1} Z_d(2n+dq; q),$$

and we conclude that

$$(61) \quad G_d(x; q) = \sum_{n \geq 0} I_d(n; q) \frac{x^{2n+dq}}{(n+dq)!n!}$$

$$(62) \quad = \sum_{n \geq 0} \binom{2n+dq}{n}^{-1} Z_d(2n+dq; q) \frac{x^{2n+dq}}{(n+dq)!n!}$$

$$(63) \quad = \sum_{n \geq 0} Z_d(2n+dq; q) \frac{x^{2n+dq}}{(2n+dq)!}$$

$$(64) \quad = \sum_{N \geq 0} Z_d(N; q) \frac{x^N}{N!}.$$

4. ASYMPTOTICS

Let us now extract an explicit formula for $Z_d(N; q)$ in terms of random contractions from Theorem 1.1. We have

$$(65) \quad G_d(x; q) = \frac{x^{dq}}{H_{d \times q}} \int_{\mathbb{B}_d} e^{x \operatorname{Tr}(P+P^*)} \gamma_d^{(q)}(dP)$$

$$(66) \quad = \sum_{N \geq 0} \frac{x^{dq}}{H_{d \times q}} \int_{\mathbb{B}_d} (\operatorname{Tr}(P+P^*))^N \gamma_d^{(q)}(dP) \frac{x^N}{N!}$$

$$(67) \quad = \sum_{n \geq 0} \frac{\binom{2n}{n}}{H_{d \times q}} \int_{\mathbb{B}_d} |\operatorname{Tr} P|^{2n} \gamma_d^{(q)}(dP) \frac{x^{2n+dq}}{(2n)!}, \text{ by (44)}$$

$$(68) \quad = \sum_{n \geq 0} \frac{(2n+dq)!}{H_{d \times q} n! n!} \int_{\mathbb{B}_d} |\operatorname{Tr} P|^{2n} \gamma_d^{(q)}(dP) \frac{x^{2n+dq}}{(2n+dq)!}.$$

Thus

$$(69) \quad Z_d(N; q) = \left[\frac{x^N}{N!} \right] G_d(x; q) = \begin{cases} \frac{(2n+dq)!}{H_{d \times q} n! n!} \int_{\mathbb{B}_d} |\operatorname{Tr} P|^{2n} \gamma_d^{(q)}(dP), & \text{if } N = 2n + dq \\ 0, & \text{otherwise} \end{cases}.$$

From this expression, we can easily obtain an asymptotic form for $Z_d(2n+dq; q)$ in the large q limit with d, n fixed. This corresponds to the “walk to infinity” with bounded backtracking Fisher’s model.

Recall the following classical result of E. Borel [6]. Let $p = (p_1, \dots, p_d, p_{d+1}, \dots, p_{d+q})$ be a uniformly random point from the real sphere $\mathbb{S}^{d+q-1} \subset \mathbb{R}^{d+q}$. Then, as $q \rightarrow \infty$, $\sqrt{q}p_1, \dots, \sqrt{q}p_d$ converge weakly to an independent family of standard Gaussian random variables. There is an ample generalization of this result to the setting of truncated random unitary matrices, first established by Petz and Réffy in [31].

Theorem 4.1. *Let $P_d^{(q)}$ be a random matrix from $\text{CUE}^{(q)}$. As $q \rightarrow \infty$, the rescaled random matrix $\sqrt{q}P_d^{(q)}$ converges to the random matrix Γ_d whose entries are i.i.d. standard complex Gaussians (in the sense of pointwise convergence of spectral correlation functions).*

We remark that this result of Petz and Réffy follows readily from the work of Sommers and Zyczkowski [35]. In particular, one knows from [35] the the spectrum

of a random matrix $P_d^{(q)}$ is a determinantal point process in the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$ governed by the Sommers-Zyczkowski kernel

$$(70) \quad \text{SZ}_d^{(q)}(z, w) = \frac{q}{\pi} (1 - |z|^2)^{\frac{q-1}{2}} (1 - |w|^2)^{\frac{q-1}{2}} \sum_{j=0}^{d-1} \binom{q+j}{j} z^j \bar{w}^j,$$

for any $q \geq 1$. One then observes that the re-scaled kernel $q^{-1} \text{SZ}_d^{(q)}(q^{-\frac{1}{2}}z, q^{-\frac{1}{2}}w)$ converges to the Ginibre kernel

$$(71) \quad \text{Gin}_d(z, w) = \frac{1}{\pi} e^{-(|z|^2 + |w|^2)} \sum_{j=0}^{d-1} \frac{1}{j!} z^j \bar{w}^j$$

as $q \rightarrow \infty$. An elementary geometric proof of Theorem 4.1 can be found in [26], while a stronger assertion has recently been proved by Krishnapur [25].

Theorem 4.1 immediately implies that

$$(72) \quad \int_{\mathbb{B}_d} |\text{Tr } P|^{2n} \gamma_d^{(q)}(dP) \sim \frac{\mathbb{E}[|z_1 + \dots + z_d|^{2n}]}{q^n}$$

as $q \rightarrow \infty$, where z_1, \dots, z_n are independent standard complex Gaussians and \mathbb{E} denotes expected value. Since the moment sequence of a standard complex Gaussian z is well-known to be

$$(73) \quad \mathbb{E}[z^m \bar{z}^n] = \delta_{m,n} n!,$$

and $z_1 + \dots + z_d$ is a complex Gaussian of variance d , we have

$$(74) \quad \int_{\mathbb{B}_d} |\text{Tr } P|^{2n} \gamma_d^{(q)}(dP) \sim \frac{d^n n!}{q^n}$$

as $q \rightarrow \infty$. Applying this asymptotic form in (69), it follows from Stirling's approximation that

$$(75) \quad Z_d(N; q) \sim (2\pi)^{\frac{1-d}{2}} \left(\prod_{i=0}^{d-1} i! \right) d^{3n+dq+\frac{1}{2}} q^{n+\frac{1-d^2}{2}}$$

as $q \rightarrow \infty$, where $N = 2n + dq$ and $n \geq 0, d \geq 1$ are fixed but arbitrary. This asymptotic form reveals an interesting phase transition in the large q asymptotic behaviour of $Z_d(N; q)$ at $n \approx \frac{d^2-1}{2}$ from

$$(76) \quad \frac{\text{exponential in } q}{\text{polynomial in } q}$$

to

$$(77) \quad (\text{exponential in } q)(\text{polynomial in } q).$$

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